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6 *j*-symbols for symmetric representations of *SO*(*n*) as the double series

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Abstract

The corrected triple sum expression of Ališauskas (1987 *J. Phys. A: Math. Gen.* **20** 35) for the recoupling (Racah) coefficients (6*j*-symbols) of the symmetric (most degenerate) representations of the orthogonal groups $SO(n)$ (previously derived from the fourfold sum expression of Ališauskas also related to the result of Hormeß and Junker (1999 *J. Phys. A: Math. Gen.* **32** 4249) is rearranged into three new different double sum expressions (related to the hypergeometric Kampé de Fériet type series) and a new triple sum expression with preferable summation condition. The Regge type symmetry of special 6*j*-symbols of the orthogonal groups $SO(n)$ in terms of special Kampé de Fériet $F_{1:3}^{1:4}$ series is revealed. The recoupling coefficients for antisymmetric representations of symplectic group $Sp(2n)$ are derived using their relation with the recoupling coefficients of the formal orthogonal group $SO(-2n)$.

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1. Introduction

The importance of 6*j* (Racah) coefficients of $SU(2)$ for the quantum angular momentum theory is well known, as well as their applications in many branches of mathematical physics, representation theory of Lie and quantum groups, in theory of orthogonal polynomials and other special functions. The Racah coefficients (6*j*-symbols) and other recoupling coefficients of the unitary $SU(n)$, orthogonal $SO(n)$ and symplectic $Sp(2n)$ groups of different rank are useful when calculating energy levels and transition rates in atomic, molecular and nuclear theory (for example, in connection with the Jahn–Teller effect and structural analysis of atomic shells, see the many papers of Judd and co-workers [1–7], for description of multi-fermionic systems and in the microscopic nuclear theory [8–14]) and in conformal field theory [15].

Special classes of coupling coefficients and 6*j*-symbols of the $SO(n)$ groups were considered by Ališauskas [16, 17], Junker and Hormeß [18, 19], with the fourfold [16, 19] and triple [16] sum expressions for the recoupling coefficients with all most degenerate (symmetric or class-one) irreducible representations. (Such $6j$-symbols have application in statistical physics, in the high-temperature expansion of the $SO(n)$-symmetric classical lattice models [18–21].) Other special expressions for $6j$-symbols of the $SO(n)$ were also considered in [22–25] and extended to the Racah coefficients of the quantum algebras $\mathrm{O}_q(n)$ [26].

The fourfold sum expressions (5.1)–(5.3) of [16] and (2)–(10) of [19] (cf the integral representation in section 6 of [18]) for the $6j$-symbols of $SO(n)$ with all six irreducible representations (irreps) symmetric are equivalent, taking into account the different expressions (11)–(15) of [19] and (3.10a)–(3.10b) of [17] for the integrals involving triplets of the Gegenbauer polynomials in terms of the very well-poised ${}_7F_6(1)$ or balanced ${}_4F_3(1)$ hypergeometric series, related to the $6j$ coefficients of $SU(2)$. The Biedenharn–Elliott identity [27, 28] (see [29–31]), used in two stages and related expansions allowed us [16] to derive the triple sum expression (5.7) for the corresponding $6j$-symbols of $SO(n)$. Note that the phase factor $(-1)^{(g-e)/2}$ (where $g \geqslant e$) should be omitted in the right-hand side of this expression, in contrast with (5.5) of the same paper.

Expressions (5.3) and (5.7) of [16] for the $6j$-symbols of $SO(n)$ are given as expansions in terms of three and two multiplied $6j$ coefficients of $SU(2)$ (with some multiple of $1/4$ parameters for odd n), respectively. The corresponding sums over the angular momentum type parameters resemble the usual expansions [29, 30] of $9j$ and $12j$ coefficients of $SU(2)$ in terms of $6j$ coefficients, which recently were rearranged by Rosengren [32] (for the $SU(1,1)$ group) and Ališauskas [33, 34] using the appropriate (less symmetric) expressions (29.1b) and (29.1c) of Jucys and Bandzaitis [29] (see also (5) and (6) in section 9.2 of [30]) for the Racah coefficients (related to the balanced hypergeometric ${}_4F_3(1)$ series) and Dougall's summation formula [35] of the very well-poised ${}_4F_3(-1)$ series. In [34], Dougall's summation formula [35, 36] of the very well-poised hypergeometric ${}_5F_4(1)$ series, together with the corresponding expressions for the Racah coefficients, was suitable for rearrangement of $12j$ coefficients of $SU(2)$. This way the total number of sums in expressions was reduced.

In this paper, the triple sum expression (5.7) of [16] for the $6j$-symbols of $SO(n)$ with all six irreps symmetric is rearranged in a similar manner into the different double sum expressions of the hypergeometric (Kampé de Fériet [37, 38]) type, as well as into the triple sum expression, with all three separate sums of the balanced ${}_4F_3(1)$ type restricted by a single parameter.

In section 2, the main results of [16] concerning the $6j$-symbols of $SO(n)$ are summarized and reconsidered in view of our objectives and some approaches used in [33, 34] in the case of $9j$ and $12j$ coefficients of $SU(2)$. Three new double sum expressions for the renormalized $6j$-symbols of $SO(n)$ (specified in terms of the so-called α-graphs $I_n(a, b, e|d, c, f)$ or related rational $c^{(\alpha,n)}_{a,b,e;d,c,f}$ functions of [19]) are derived in section 3, where the Regge [39] type symmetry is also revealed (for $n \geqslant 5$), as well as the role of the Bargmann–Shelepin [40, 41] parameters, extended from the $6j$ coefficients of $SU(2)$ or $SO(3)$. The triple sum expression for the renormalized $6j$-symbols of $SO(n)$ presented in section 4 sometimes may be more preferable, similarly as special expressions of the stretched or almost stretched $6j$-symbols of $SO(n)$.

In section 5, the renormalized $6j$-symbols of $SO(n)$ are expanded in terms of (numerator) Pochhammer symbols, as well as in terms of the special class of Kampé de Fériet [37, 38]) functions $F^{1:4;4}_{1:3;3}[{}^{\cdots}_{\cdots}; 1, 1]$, whose specific features and diversity are considered.

The recoupling coefficients for antisymmetric representations $\langle 1^{\nu}\rangle$ of the symplectic group $Sp(2n)$ are presented in the appendix as a formal analytical continuation of the recoupling coefficients for symmetric representations of the orthogonal group with negative rank $SO(-2n)$, in accordance with [3, 7, 22] (cf also [42, 43]).

2. Preliminaries

In accordance with (5.3) of [16], we may express the $6j$-symbol of $SO(n)$ $(n \geqslant 4)$ with all representations symmetric as follows:

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{SO(n)} = \left[\frac{(2c+n-2)(2d+n-2)(2e+n-2)}{8d_c^{(n)} d_d^{(n)} d_e^{(n)}} \right]^{1/2}$$
$$\times \begin{pmatrix} c & d & e \\ 0 & 0 & 0 \end{pmatrix}_n^{-1} \sum_{l'} (-1)^{(c+d-e)/2+f+n+l'} (2l'+n-3)$$
$$\times \begin{Bmatrix} \frac{1}{2}b & \frac{1}{2}f+\frac{1}{4}n-1 & \frac{1}{2}d+\frac{1}{4}n-1 \\ \frac{1}{2}f+\frac{1}{4}n-1 & \frac{1}{2}(b+n)-2 & l'+\frac{1}{2}n-2 \end{Bmatrix}$$
$$\times \begin{Bmatrix} \frac{1}{2}a & \frac{1}{2}f+\frac{1}{4}n-1 & \frac{1}{2}c+\frac{1}{4}n-1 \\ \frac{1}{2}f+\frac{1}{4}n-1 & \frac{1}{2}(a+n)-2 & l'+\frac{1}{2}n-2 \end{Bmatrix}$$
$$\times \begin{Bmatrix} \frac{1}{2}a & \frac{1}{2}b+\frac{1}{4}n-1 & \frac{1}{2}e+\frac{1}{4}n-1 \\ \frac{1}{2}b+\frac{1}{4}n-1 & \frac{1}{2}(a+n)-2 & l'+\frac{1}{2}n-2 \end{Bmatrix} \left[\frac{l'!(n-3)!}{(l'+n-4)!} \right]^{1/2} \quad (2.1)$$

where in the right-hand side the usual $6j$ coefficients of $SU(2)$ [29–31] appear for n even. Otherwise, for n odd some integer linear combination of parameters of the type $a-l'$, l' or $(b+d-f)/2$ are also restricting the summation intervals in extensions of the asymmetric (Jucys–Bandzaitis) expressions for $6j$ coefficients (as presented by (2.1a), (2.1b), (2.2a) and (2.2b) in [33] for $q=1$), with some ratios of factorials $x!/y!$ turning into ratios of the gamma functions $\Gamma(x+1)/\Gamma(y+1)$ with half-integer arguments.

The dimension

$$d_l^{(n)} = \frac{(2l+n-2)(l+n-3)!}{l!(n-2)!} \quad (2.2)$$

of the $SO(n)$ symmetric irreducible representation l and special $3j$-symbols

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_n = (-1)^{\psi_n} \frac{1}{\Gamma(n/2)} \left[\frac{(J+n-3)!}{(n-3)!\Gamma(J+n/2)} \right.$$
$$\left. \times \prod_{i=1}^{3} \frac{(l_i+n/2-1)\,\Gamma\,(J-l_i+n/2-1)}{d_{l_i}^{(n)}(J-l_i)!} \right]^{1/2} \quad (2.3a)$$
$$= (-1)^{\psi_n} \frac{\widetilde{\nabla}_{n[0,1,2,3]}^{-1}(l_1,l_2,l_3)}{\Gamma(n/2)} \left[\frac{1}{(n-3)!} \prod_{i=1}^{3} \frac{l_i+n/2-1}{d_{l_i}^{(n)}} \right]^{1/2} \quad (2.3b)$$

(see [16–18] and related special Clebsch–Gordan coefficients [44, 45]), used in (2.1) and further, are rational numbers (in part under the square root sign). In (2.3a), $J = \frac{1}{2}(l_1+l_2+l_3)$ and $J-l_i$ $(i=1,2,3)$ are non-negative integers. The triangular coefficient $\widetilde{\nabla}_{n[0,1,2,3]}(l_1,l_2,l_3)$ in (2.3b) may be expressed as follows

$$\widetilde{\nabla}_{n[0,1,2,3]}(a,b,e) = \left(\frac{\left[\frac{1}{2}(b+e-a)\right]!\left[\frac{1}{2}(a-b+e)\right]!}{\Gamma\left(\frac{1}{2}(b+e-a+n)-1\right)\Gamma\left(\frac{1}{2}(a-b+e+n)-1\right)} \right.$$
$$\left. \times \frac{\left[\frac{1}{2}(a+b-e)\right]!\Gamma\left(\frac{1}{2}(a+b+e+n)\right)}{\Gamma\left(\frac{1}{2}(a+b-e+n)-1\right)\left[\frac{1}{2}(a+b+e)+n-3\right]!} \right)^{1/2}. \quad (2.4)$$

It is reasonable to take $\psi_n = 0$ for $n \geqslant 4$ (cf [16, 17, 45]).

Six (from 24) elementary symmetry properties of the $6j$-symbols of $SO(n)$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}_{SO(n)} = \begin{Bmatrix} j_a & j_b & j_c \\ l_a & l_b & l_c \end{Bmatrix}_{SO(n)} = \begin{Bmatrix} j_a & l_b & l_c \\ l_a & j_b & j_c \end{Bmatrix}_{SO(n)} \tag{2.5}$$

are visible from expression (2.1) (see also (16) of [19]).

We may replace the last two factors (in the last line) on the right-hand side of (2.1) by

$$\begin{aligned} &\widetilde{\nabla}_{n[0,3,5,6]}(a, b; e, 0) \sum_{g \geqslant e} \frac{(g+n-3)\Gamma\left(\frac{1}{2}(g-e+n)-2\right)\left[\frac{1}{2}(g+e)+n-4\right]!}{\left[\frac{1}{2}(g-e)\right]!\Gamma\left(\frac{1}{2}(g+e+n)\right)\Gamma\left(\frac{1}{2}n-2\right)} \\ &\quad \times (-1)^{(g-e)/2} \left(\frac{(n-3)!\left[\frac{1}{2}(a-b+g)\right]!\left[\frac{1}{2}(b-a+g)\right]!}{\left[\frac{1}{2}(a-b+g)+n-4\right]!\left[\frac{1}{2}(b-a+g)+n-4\right]!} \right)^{1/2} \\ &\quad \times \begin{Bmatrix} \frac{1}{2}(a+n)-2 & \frac{1}{2}a & l'+\frac{1}{2}n-2 \\ \frac{1}{2}(b+n)-2 & \frac{1}{2}b & \frac{1}{2}(g+n)-2 \end{Bmatrix}, \end{aligned} \tag{2.6}$$

valid also in accordance with the $q = 1$ version of expression (2.1b) of [33] and Dougall's summation formula (2.3.4.5) of [35] for special very well-poised ${}_5F_4(1)$ series as presented by an extension of (A1a) of [34] (after replacing $\Gamma(-x)/\Gamma(-y)$ if necessary by $(-1)^{x-y}\Gamma(y+1)/\Gamma(x+1)$ for $x - y$ integer) with parameters

$$\begin{aligned} &j \to \tfrac{1}{2}(g+n)-2 \qquad p_1 \to -\tfrac{1}{2}(e+n)+1 \qquad p_2 \to \tfrac{1}{2}e \\ &p_3 \to -\tfrac{1}{2}(a+b+n) \qquad p_4 \to \tfrac{1}{2}(a+b+n)-2-l'+z \end{aligned}$$

and integer $p_1 + p_4 + 1$, restricting the summation interval of ${}_5F_4(1)$ series. Another triangular coefficient

$$\begin{aligned} \widetilde{\nabla}_{n[0,3,5,6]}(a, b; e, 0) = &\left(\frac{\Gamma\left(\frac{1}{2}(b+e-a+n)-1\right)\Gamma\left(\frac{1}{2}(a-b+e+n)-1\right)}{\left[\frac{1}{2}(b+e-a)\right]!\left[\frac{1}{2}(a-b+e)\right]!} \right. \\ &\left. \times \frac{\left[\frac{1}{2}(a+b-e)\right]!\Gamma\left(\frac{1}{2}(a+b+e+n)\right)}{\Gamma\left(\frac{1}{2}(a+b-e+n)-1\right)\left[\frac{1}{2}(a+b+e)+n-3\right]!} \right)^{1/2} \end{aligned} \tag{2.7}$$

in (2.6) is related to $\nabla_{n[0,3,5,6]}(a, b; e, 0)$ as defined by (2.3) of [16], but coincides with it only for even n.

Hence, the Biedenharn–Elliott identity, applied to the triplet of $6j$-coefficients of $SU(2)$ in (2.1) with substituted (2.6), allowed us to present the special $6j$-symbol of $SO(n)$ (cf (5.7) of [16]) for $n > 4$ as follows:

$$\begin{aligned} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{SO(n)} = &\left[\frac{(2c+n-2)(2d+n-2)(2e+n-2)}{8d_c^{(n)}d_d^{(n)}d_e^{(n)}} \right]^{1/2} \begin{pmatrix} c & d & e \\ 0 & 0 & 0 \end{pmatrix}_n^{-1} \\ &\times \widetilde{\nabla}_{n[0,3,5,6]}(a, b; e, 0) \sum_{g=e}^{a+b} \frac{(g+n-3)\Gamma\left(\frac{1}{2}(g-e+n)-2\right)}{\left[\frac{1}{2}(g-e)\right]!\Gamma\left(\frac{1}{2}(g+e+n)\right)\Gamma\left(\frac{1}{2}n-2\right)} \\ &\times \left(\frac{\left[\frac{1}{2}(a-b+g)\right]!\left[\frac{1}{2}(b-a+g)\right]!(n-3)!}{\left[\frac{1}{2}(a-b+g)+n-4\right]!\left[\frac{1}{2}(b-a+g)+n-4\right]!} \right)^{1/2} \\ &\times \left[\tfrac{1}{2}(g+e)+n-4\right]! \begin{Bmatrix} \frac{1}{2}c+\frac{1}{4}n-1 & \frac{1}{2}a & \frac{1}{2}f+\frac{1}{4}n-1 \\ \frac{1}{2}(b+n)-2 & \frac{1}{2}d+\frac{1}{4}n-1 & \frac{1}{2}(g+n)-2 \end{Bmatrix} \\ &\times \begin{Bmatrix} \frac{1}{2}b & \frac{1}{2}(a+n)-2 & \frac{1}{2}(g+n)-2 \\ \frac{1}{2}c+\frac{1}{4}n-1 & \frac{1}{2}d+\frac{1}{4}n-1 & \frac{1}{2}f+\frac{1}{4}n-1 \end{Bmatrix} \end{aligned} \tag{2.8}$$

with the product of two $6j$-coefficients of $SU(2)$ (some parameters of which accept values in multiples of $1/4$ for odd n) in the right-hand side. It was suggested in [16] to use for them the most symmetric (Racah) expression (see [29–31]), useless, however, for the rearrangement of (2.8).

3. Double sum expressions for $6j$-symbols of $SO(n)$

Nevertheless, we may rearrange (2.8) expressing the second $6j$-coefficient of $SU(2)$ by means of (2.1a) of [33] and the first one by means of (2.1b) of [33]. In this case the factors, depending on the summation parameter $j = (g+n)/2 - 2$ and distributed in the numerators and denominators of different $6j$-coefficients under the square root, cancel, together with the asymmetric triangle coefficients

$$\nabla(xyj) = \left[\frac{(x+y-j)!(x-y+j)!(x+y+j+1)!}{(y+j-x)!}\right]^{1/2} \tag{3.1a}$$

$$= \left[\frac{\Gamma(x+y-j+1)\Gamma(x-y+j+1)\Gamma(x+y+j+2)}{\Gamma(y+j-x+1)}\right]^{1/2}. \tag{3.1b}$$

Then we again may use Dougall's summation formula for very well-poised ${}_5F_4(1)$ series as presented by (A1b) of [34] with parameters

$$j \to \tfrac{1}{2}(g+n) - 2 \qquad p_1 \to -\tfrac{1}{2}(e+n) + 1 \qquad p_2 \to \tfrac{1}{2}e$$
$$p_3 \to \tfrac{1}{2}(f-b-c) - 1 + z_2 \qquad p_4 \to \tfrac{1}{2}(b+c-f+n) - 2 + z_1.$$

Two more different rearrangements of (2.8) are also possible in the following ways: the second version may be obtained when we express the last $6j$-coefficients of $SU(2)$ on the right-hand side

$$\begin{Bmatrix} \frac{1}{2}(a+n)-2 & \frac{1}{2}f+\frac{1}{4}n-1 & \frac{1}{2}c+\frac{1}{4}n-1 \\ \frac{1}{2}d+\frac{1}{4}n-1 & \frac{1}{2}(g+n)-2 & \frac{1}{2}b \end{Bmatrix} \tag{3.2}$$

(with transposed parameters) by means of (2.2a) of [33] and use the extended version of (A1a) of [34] with parameters

$$j \to \tfrac{1}{2}(g+n) - 2 \qquad p_1 \to -\tfrac{1}{2}(e+n) + 1 \qquad p_2 \to \tfrac{1}{2}e$$
$$p_3 \to z_2 - \tfrac{1}{2}(b+c+f+n) \qquad p_4 \to \tfrac{1}{2}(b+c-f+n) - 2 + z_1.$$

The third version may be obtained when we express the $6j$-coefficients of $SU(2)$ on the right-hand side of (2.8)

$$\begin{Bmatrix} \frac{1}{2}(g+n)-2 & \frac{1}{2}a & \frac{1}{2}(b+n)-2 \\ \frac{1}{2}f+\frac{1}{4}n-1 & \frac{1}{2}d+\frac{1}{4}n-1 & \frac{1}{2}c+\frac{1}{4}n-1 \end{Bmatrix}$$
$$\times \begin{Bmatrix} \frac{1}{2}c+\frac{1}{4}n-1 & \frac{1}{2}f+\frac{1}{4}n-1 & \frac{1}{2}(a+n)-2 \\ \frac{1}{2}b & \frac{1}{2}(g+n)-2 & \frac{1}{2}d+\frac{1}{4}n-1 \end{Bmatrix} \tag{3.3}$$

by means of (2.2a) and (2.1a) of [33], respectively, and use an extended version of (A1b) of [34] with parameters

$$j \to \tfrac{1}{2}(g+n) - 2 \qquad p_1 \to -\tfrac{1}{2}(e+n) + 1 \qquad p_2 \to \tfrac{1}{2}e$$
$$p_3 \to \tfrac{1}{2}(c-d) - 1 - z_2 \qquad p_4 \to \tfrac{1}{2}(a+b+n) - 2 - z_1.$$

In all three cases the summation intervals over j (or g) are restricted by non-negative integers $p_1 + p_4 + 1$. In contrast with the case of $9j$ and $12j$ coefficients of $SU(2)$ (see [33, 34]), the formal summation intervals over g cannot exceed $\frac{1}{2}\min(a + b - e, c + d - e)$ (determined by triangular conditions) in the main and replaced by (3.2) versions of (2.8). Nevertheless, the possible superfluous terms arising for $g = c + d + 2, c + d + 4, \ldots,$ in the third version of (2.8) (with $6j$ coefficients replaced by (3.3)) are unimportant, since in this case the sum over z_2 turns into 0, in accordance with Karlsson's summation formula [36] (cf section 2 of [33]).

Now it is convenient to write the $6j$-symbol of $SO(n)$ in terms of the so-called α-graph $I_n(a, b, e|d, c, f)$ or related quantity $c^{(\alpha,n)}_{a,b,e;d,c,f}$ (see [19])

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{SO(n)} = c^{(\alpha,n)}_{a,b,e;d,c,f} \left[d^{(n)}_a d^{(n)}_b d^{(n)}_c d^{(n)}_d d^{(n)}_e d^{(n)}_f \begin{pmatrix} a & b & e \\ 0 & 0 & 0 \end{pmatrix}_n \right.$$
$$\left. \times \begin{pmatrix} a & c & f \\ 0 & 0 & 0 \end{pmatrix}_n \begin{pmatrix} b & d & f \\ 0 & 0 & 0 \end{pmatrix}_n \begin{pmatrix} c & d & e \\ 0 & 0 & 0 \end{pmatrix}_n \right]^{-1} \tag{3.4a}$$
$$= \widetilde{\nabla}_{n[0,1,2,3]}(a, b, e) \widetilde{\nabla}_{n[0,1,2,3]}(a, c, f) \widetilde{\nabla}_{n[0,1,2,3]}(b, d, f)$$
$$\times \widetilde{\nabla}_{n[0,1,2,3]}(c, d, e) \frac{[(n-3)!]^2 \Gamma^4(n/2)}{(a, b, c, d, e, f)_{[n]}} c^{(\alpha,n)}_{a,b,e;d,c,f} \tag{3.4b}$$

where

$$(a, b, c, d, e, f)_{[n]} = \tfrac{1}{64}(2a + n - 2)(2b + n - 2)(2c + n - 2)$$
$$\times (2d + n - 2)(2e + n - 2)(2f + n - 2)$$

and the quantities

$$c^{(\alpha,n)}_{a,b,e;d,c,f} = d^{(n)}_a d^{(n)}_b d^{(n)}_c d^{(n)}_d d^{(n)}_e d^{(n)}_f I_n(a, b, e|d, c, f)$$
$$= d^{(n)}_a d^{(n)}_b d^{(n)}_c d^{(n)}_d d^{(n)}_e d^{(n)}_f \int_{SO(n)} \mathrm{d}g_1 \int_{SO(n)} \mathrm{d}g_2 \int_{SO(n)} \mathrm{d}g_3 D^a_{00}(g_1)$$
$$\times D^b_{00}(g_2) D^e_{00}(g_3) D^d_{00}\left(g_2^{-1} g_3\right) D^c_{00}\left(g_3^{-1} g_1\right) D^f_{00}\left(g_1^{-1} g_2\right) \tag{3.5}$$

are rational numbers and the triangular coefficients $\widetilde{\nabla}_{n[0,1,2,3]}(l_1, l_2, l_3)$ are defined by (2.4). Here $D^l_{00}(g)$ are the zonal spherical functions [46] of irrep l of $SO(n)$. In our phase system with $\psi_n = 0$ the phase factor $(-1)^{d+e+f}$ of (2) of [19] vanishes.

From (2.8) after summation over g we obtain three following different expressions for coefficients (3.5):

$$c^{(\alpha,n)}_{a,b,e;d,c,f} = (a, b, c, d, e, f)_{[n]} \frac{\left[\frac{1}{2}(a + c + f) + n - 3\right]! \Gamma\left(\frac{1}{2}(a + c - f + n) - 1\right)}{(n-3)! \Gamma\left(\frac{1}{2}(a + c + f + n)\right) \left[\frac{1}{2}(a + c - f)\right]!}$$
$$\times \frac{\Gamma\left(\frac{1}{2}(b + e - a + n) - 1\right) \Gamma\left(\frac{1}{2}(a - b + e + n) - 1\right)}{\Gamma^3\left(\frac{1}{2}n\right) \left[\frac{1}{2}(b + e - a)\right]! \left[\frac{1}{2}(a - b + e)\right]!} (-1)^{(b+c-e-f)/2}$$
$$\times \sum_{z_1, z_2} \frac{(-1)^{z_1+z_2} \Gamma\left(\frac{1}{2}(b + d - f + n) - 1 + z_1\right) \left[\frac{1}{2}(a + c - f) + z_1\right]!}{z_1! \left[\frac{1}{2}(a - c + f) - z_1\right]! \left[\frac{1}{2}(d + f - b) - z_1\right]!}$$
$$\times \frac{\Gamma\left(f + \frac{1}{2}n - 1 - z_1\right) \Gamma\left(\frac{1}{2}(b + c + e - f + n) - 1 - z_2\right)}{\left[\frac{1}{2}(b + c - e - f) + z_1\right]! \Gamma\left(\frac{1}{2}(b + c + e - f + n) + z_1\right) z_2!}$$

$$\times \frac{\Gamma\left(\frac{1}{2}(d+f-b+n)-1+z_2\right)\Gamma\left(\frac{1}{2}(a-c+f+n)-1+z_2\right)}{\left[\frac{1}{2}(b+d-f)-z_2\right]!\,\Gamma\left(\frac{1}{2}(a+c-f+n)-1-z_2\right)}$$

$$\times \frac{(z_1+z_2)!}{\left[\frac{1}{2}(e+f-b-c)+z_2\right]!\,\Gamma\left(f+\frac{1}{2}n+z_2\right)\Gamma\left(\frac{1}{2}n-1+z_1+z_2\right)} \tag{3.6a}$$

$$= (a,b,c,d,e,f)_{[n]}\frac{\left[\frac{1}{2}(a+c+f)+n-3\right]!\,\Gamma\left(\frac{1}{2}(a+c-f+n)-1\right)}{(n-3)!\,\Gamma\left(\frac{1}{2}(a+c+f+n)\right)\left[\frac{1}{2}(a+c-f)\right]!}$$

$$\times \frac{\Gamma\left(\frac{1}{2}(b+e-a+n)-1\right)\Gamma\left(\frac{1}{2}(a-b+e+n)-1\right)}{\Gamma^3\left(\frac{1}{2}n\right)\left[\frac{1}{2}(b+e-a)\right]!\left[\frac{1}{2}(a-b+e)\right]!}(-1)^{(a-b-c+d)/2}$$

$$\times \sum_{z_1,z_2}\frac{(-1)^{z_1+z_2}\Gamma\left(\frac{1}{2}(b+d-f+n)-1+z_1\right)\left[\frac{1}{2}(a+c-f)+z_1\right]!}{z_1!z_2!\left[\frac{1}{2}(a-c+f)-z_1\right]!\left[\frac{1}{2}(b+c-e-f)+z_1\right]!}$$

$$\times \frac{\Gamma\left(f+\frac{1}{2}n-1-z_1\right)(f-z_1-z_2)!}{\left[\frac{1}{2}(d+f-b)-z_1\right]!\,\Gamma\left(\frac{1}{2}(b+c+e-f+n)+z_1\right)}$$

$$\times \frac{\Gamma\left(\frac{1}{2}(b+c-e+f+n)-1-z_2\right)}{\Gamma\left(f+\frac{1}{2}n-1-z_1-z_2\right)\left[\frac{1}{2}(b-d+f)-z_2\right]!\left[\frac{1}{2}(c+f-a)-z_2\right]!}$$

$$\times \frac{\Gamma\left(f+\frac{1}{2}n-1-z_2\right)\left[\frac{1}{2}(b+c+e+f)+n-3-z_2\right]!}{\Gamma\left(\frac{1}{2}(b+d+f+n)-z_2\right)\left[\frac{1}{2}(a+c+f)+n-3-z_2\right]!} \tag{3.6b}$$

$$= (a,b,c,d,e,f)_{[n]}\frac{\Gamma\left(\frac{1}{2}(c+f-a+n)-1\right)\Gamma\left(\frac{1}{2}(a-c+f+n)-1\right)}{(n-3)!\left[\frac{1}{2}(c+f-a)\right]!\left[\frac{1}{2}(a-c+f)\right]!}$$

$$\times \frac{\Gamma\left(\frac{1}{2}(b+e-a+n)-1\right)\Gamma\left(\frac{1}{2}(a-b+e+n)-1\right)}{\Gamma^3\left(\frac{1}{2}n\right)\left[\frac{1}{2}(b+e-a)\right]!\left[\frac{1}{2}(a-b+e)\right]!}(-1)^{(a+d-e-f)/2}$$

$$\times \sum_{z_1,z_2}\frac{(-1)^{z_1+z_2}\Gamma\left(\frac{1}{2}(a+b+c-d+n)-1-z_1\right)(a-z_1)!}{z_1!z_2!\left[\frac{1}{2}(a+b-e)-z_1\right]!\,\Gamma\left(\frac{1}{2}(a+b+e+n)-z_1\right)}$$

$$\times \frac{\left[\frac{1}{2}(a+b+c+d)+n-3-z_1\right]!\,\Gamma\left(\frac{1}{2}(d+f-b+n)-1+z_2\right)}{\left[\frac{1}{2}(a+c-f)-z_1\right]!\,\Gamma\left(\frac{1}{2}(a+c+f+n)-z_1\right)\left[\frac{1}{2}(b-d+f)-z_2\right]!}$$

$$\times \frac{\Gamma\left(\frac{1}{2}(d+e-c+n)-1+z_2\right)}{\left[\frac{1}{2}(c-d+e)-z_2\right]!\,\Gamma\left(\frac{1}{2}(a-b-c+d+n)-1+z_2\right)}$$

$$\times \frac{\Gamma\left(\frac{1}{2}(a+b+c-d+n)-1-z_2\right)\left[\frac{1}{2}(a+b+c-d)-z_1-z_2\right]!}{\Gamma\left(d+\frac{1}{2}n+z_2\right)\Gamma\left(\frac{1}{2}(a+b+c-d+n)-1-z_1-z_2\right)} \tag{3.6c}$$

without the visible symmetry properties of 6 j-symbols of the orthogonal $SO(n)$ group. These expressions are valid for $n \geqslant 4$ (and probably for $c^{(\alpha,3)}_{a,b,e;d,c,f}$). When $n = 4$, the numerator and denominator factorials (gamma functions) depending on z_1+z_2 cancel and 6 j-symbols of $SO(4)$ split into the product (in this case equal to the square) of two 6 j coefficients of $SU(2)$.

All separate sums over z_1 or z_2 in (3.6a)–(3.6c) correspond to the terminating balanced (Saalschützian) ${}_5F_4(1)$ series (cf [35, 36]), with summation intervals restricted by

$$\tfrac{1}{2}\min(a-c+f, d+f-b) \qquad \text{and} \qquad \tfrac{1}{2}(d+e-c) \qquad \text{for} \quad \tfrac{1}{2}(b+c-e-f) \geqslant 0 \tag{3.7a}$$

or by

$$\tfrac{1}{2}\min(a+b-e, c+d-e) \qquad \text{and} \qquad \tfrac{1}{2}(b+d-f) \qquad \text{for} \quad \tfrac{1}{2}(b+c-e-f) \leqslant 0 \tag{3.7b}$$

in (3.6*a*), by

$$\tfrac{1}{2}\min(a-c+f, d+f-b, a+b-e, c+d-e) \qquad \text{and} \qquad \tfrac{1}{2}\min(b-d+f, c+f-a) \tag{3.7c}$$

in (3.6*b*) and by

$$\tfrac{1}{2}\min(a+b-e, a+c-f) \qquad \text{and} \qquad \tfrac{1}{2}\min(b-d+f, c-d+e) \tag{3.7d}$$

in (3.6*c*).

Using (3.4*b*), together with expression (3.6*a*) or (3.6*b*) for coefficients $c^{(\alpha,n)}_{a,b,e;d,c,f}$ (after cancelling the dimensions of irreps and the $(a, b, c, d, e, f)_{[n]}$ type factors), the Regge-type symmetry (cf [39])

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{SO(n)} = \begin{Bmatrix} s_3-a & s_3-b & e \\ s_3-d & s_3-c & f \end{Bmatrix}_{SO(n)} \tag{3.8}$$

of special $6j$-symbols of $SO(n)$ (where $s_3 = \frac{1}{2}(a+b+c+d)$) may be checked. Otherwise, using (3.4*b*), together with expression (3.6*c*), the usual and Regge-type symmetries

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{SO(n)} = \begin{Bmatrix} a & s_1-e & s_1-b \\ d & s_1-f & s_1-c \end{Bmatrix}_{SO(n)} \tag{3.9a}$$

$$= \begin{Bmatrix} a & s_1-f & s_1-c \\ d & s_1-e & s_1-b \end{Bmatrix}_{SO(n)} = \begin{Bmatrix} a & c & f \\ d & b & e \end{Bmatrix}_{SO(n)} \tag{3.9b}$$

where $s_1 = \frac{1}{2}(b+c+e+f)$, are visible. The symmetries (3.8) and (3.9*a*) correspond to some column transpositions of Shelepin's [41] $4 \times 3R$-array

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = \begin{vmatrix} a+b-e & a+c-f & b+d-f & c+d-e \\ a-c+f & a-b+e & d+e-c & d+f-b \\ b-d+f & c-d+e & b+e-a & c+f-a \end{vmatrix} \tag{3.10a}$$

$$\times \begin{vmatrix} 2r_{11} & 2r_{12} & 2r_{13} & 2r_{14} \\ 2r_{21} & 2r_{22} & 2r_{23} & 2r_{24} \\ 2r_{31} & 2r_{32} & 2r_{33} & 2r_{34} \end{vmatrix} \tag{3.10b}$$

of $6j$ coefficients (cf (29.32) of [29] or (12) in section 9.1 of [30]). Array (3.10*a*) (cf also [40]) is also convenient for the description of 144 symmetries of $6j$-symbols of $SO(n)$ under arbitrary transpositions of its columns or rows. Since all the entries of (3.10*a*) are even integers for $n \geqslant 4$, integer parameters $r_{ik} = \beta_i - \alpha_k$ ($i = 1, 2, 3$; $j = 1, 2, 3, 4$) may be more convenient, as well as the most symmetric parametrization

$$\begin{aligned} &\alpha_1 = \tfrac{1}{2}(c+d+e) && \alpha_2 = \tfrac{1}{2}(b+d+f) && \alpha_3 = \tfrac{1}{2}(a+c+f) \\ &\alpha_4 = \tfrac{1}{2}(a+b+e) && \beta_1 = \tfrac{1}{2}(a+b+c+d) && \\ &\beta_2 = \tfrac{1}{2}(a+d+e+f) && \beta_3 = \tfrac{1}{2}(b+c+e+f) && \end{aligned} \tag{3.11}$$

with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \beta_1 + \beta_2 + \beta_3$ (cf [29]).

Expression (3.6*a*) includes the minimum of terms, when minimal values are accepted by the parameters in the same (the first or the second) row of array (3.10*a*) (r_{i3} and r_{i1} or r_{i4}, for $i = 1$ or 2). Otherwise, (3.6*b*) and (3.6*c*) include the minimum of terms, when minimal values are accepted by the parameters in the same column (respectively, r_{i1} and r_{31} or r_{i4} and

r_{34}, $i = 1$ or 2 for (3.6b), or r_{11} and r_{31} or r_{14} and r_{34} in the last case). As a rule (with a single exception in each case), the definite triplets of numerator and denominator factorials or gamma functions, depending on summation parameters z_1, z_2 and $z_1 + z_2$, form in (3.6a), (3.6b) and (3.6c) the binomial coefficients (e.g., $z_1!$, $z_2!$ and $(z_1 + z_2)!$), their analytical continuation or beta functions which respond to relations (5.9a) or (5.9b) between the parameters of special Kampé de Fériet [37, 38] functions $F^{1:4;4}_{1:3;3}[{}^{\cdots}_{\cdots}; 1, 1]$, considered in section 5.

4. Expressions for $6j$-symbols of $SO(n)$ with summation restricted by single parameter

The double sum expressions of $6j$-symbols of $SO(n)$ may be inconvenient, when $r_{11} \ll r_{1k}$ $(k = 2, 3, 4)$ and $r_{11} \ll r_{i1}$ $(i = 2, 3)$. In the stretched case of $6j$-symbols of $SO(n)$ (with $r_{11} = 0$) we obtain

$$c^{(\alpha,n)}_{a,b,a+b;d,c,f} = \frac{(a,b,c,d,e,f)_{[n]}\Gamma\left(a+\frac{1}{2}n-1\right)\Gamma\left(b+\frac{1}{2}n-1\right)}{(n-3)!\Gamma^3\left(\frac{1}{2}n\right)\nabla^2\left(\frac{1}{2}a,\frac{1}{2}c+\frac{1}{4}n-1,\frac{1}{2}f+\frac{1}{4}n-1\right)} \times \frac{\nabla^2\left(\frac{1}{2}(e+n)-2,\frac{1}{2}c+\frac{1}{4}n-1,\frac{1}{2}d+\frac{1}{4}n-1\right)}{\nabla^2\left(\frac{1}{2}b,\frac{1}{2}d+\frac{1}{4}n-1,\frac{1}{2}f+\frac{1}{4}n-1\right)\Gamma\left(e+\frac{1}{2}n\right)} \qquad (4.1)$$

(cf (5.4) of [16]), since some parameter from sets (3.7a)–(3.7c) turns into 0 (possibly after using some symmetry property of $6j$-symbols), together with a fixed corresponding summation parameter, when another sum turns into the summable balanced ${}_3F_2(1)$ series (see [35, 36] and appendix A of [34]). When the vanishing linear combination of parameters of the stretched $6j$-symbol does not belong to sets (3.7a)–(3.7c) the summation of special cases of (3.6a)–(3.6c) is more difficult.

In a near to the stretched case with $e = a + b - 2$, the sum over z_1 in (3.6a) includes two terms and we have the ${}_4F_3(1)$ type sums over z_2 corresponding to the $6j$ coefficients of $SU(2)$. Using for them the most symmetric (Racah) expression (see [29–31]) we derive the following expression for special coefficients (3.5):

$$\begin{aligned} c^{(\alpha,n)}_{a,b,a+b-2;d,c,f} ={}& \frac{(a,b,c,d,e,f)_{[n]}\Gamma\left(a+\frac{1}{2}n-2\right)\Gamma\left(b+\frac{1}{2}n-2\right)}{(n-3)!\Gamma^3\left(\frac{1}{2}n\right)\nabla^2\left(\frac{1}{2}a,\frac{1}{2}c+\frac{1}{4}n-1,\frac{1}{2}f+\frac{1}{4}n-1\right)} \\ &\times \frac{\nabla^2\left(\frac{1}{2}(e+n)-2,\frac{1}{2}c+\frac{1}{4}n-1,\frac{1}{2}d+\frac{1}{4}n-1\right)}{64\Gamma\left(e+\frac{1}{2}n+1\right)\nabla^2\left(\frac{1}{2}b,\frac{1}{2}d+\frac{1}{4}n-1,\frac{1}{2}f+\frac{1}{4}n-1\right)} \\ &\times \{2a(c+d-e)(e-c+d+n-2)[(c+d-e+n-4)(b+d-f) \\ &\times (a+c-f+n-4) - (c+d+e+2n-4)(a-c+f)(b-d+f)] \\ &+ (2e+n)(a-c+f)(c+f-a+n-2)[(c+d+e+2n-4) \\ &\times (b-d+f)(a-c+f+n-4) - (b+d-f)(c+d-e) \\ &\times (a+c-f+n-4)]\}. \end{aligned} \qquad (4.2)$$

Expressions (4.1) and (4.2) cover all but the last entries of table 1 of [19].

Otherwise, all three summation intervals in expression (2.8) are restricted by $r_{11} = \frac{1}{2}(a + b - e)$. The sum over g in (2.8) turns into the very well-poised hypergeometric ${}_7F_6(1)$ series when we express the $6j$-coefficients of $SU(2)$ on the right-hand side of (2.8)

$$\begin{Bmatrix} \frac{1}{2}(b+n)-2 & \frac{1}{2}d+\frac{1}{4}n-1 & \frac{1}{2}f+\frac{1}{4}n-1 \\ \frac{1}{2}c+\frac{1}{4}n-1 & \frac{1}{2}a & \frac{1}{2}(g+n)-2 \end{Bmatrix} \times \begin{Bmatrix} \frac{1}{2}(g+n)-2 & \frac{1}{2}b & \frac{1}{2}(a+n)-2 \\ \frac{1}{2}f+\frac{1}{4}n-1 & \frac{1}{2}c+\frac{1}{4}n-1 & \frac{1}{2}d+\frac{1}{4}n-1 \end{Bmatrix} \qquad (4.3)$$

by means of (2.1a) (with inverted summation parameter) and (2.2a) of [33], respectively. Using Watson's transformation formula (2.5.1) of [36] or (6.10) of [47], we rearrange the sum over g into balanced ${}_4F_3(1)$ hypergeometric series (see also the related transition between expression (C3) for the $6j$ coefficients [48] in terms of (3.1a) and expression (2.1a) of [33]), with inverted sum. Then instead of (2.8) we obtain the following triple sum expression for coefficients (3.5):

$$
\begin{aligned}
c^{(\alpha,n)}_{a,b,e;d,c,f} &= (a,b,c,d,e,f)_{[n]} \frac{\left[\frac{1}{2}(a+b+e)+n-3\right]!\,\Gamma\left(\frac{1}{2}(a+b-e+n)-1\right)}{(n-3)!\,\Gamma\left(\frac{1}{2}(a+c+f+n)\right)\left[\frac{1}{2}(a+c-f)\right]!} \\
&\quad\times \frac{\Gamma\left(\frac{1}{2}(b+e-a+n)-1\right)\Gamma\left(\frac{1}{2}(a-b+e+n)-1\right)}{\Gamma^3\left(\frac{1}{2}n\right)\left[\frac{1}{2}(b+e-a)\right]!\left[\frac{1}{2}(a-b+e)\right]!} \\
&\quad\times \frac{\Gamma\left(\frac{1}{2}(d-b+f+n)-1\right)}{\left[\frac{1}{2}(b-d+f)\right]!} \sum_{z_1,z_2,z_3} \frac{(-1)^{(a+b-e)/2+z_1+z_2+z_3}(a-z_1)!}{z_1!z_2!z_3!\left[\frac{1}{2}(c+d-a-b)+z_1\right]!} \\
&\quad\times \frac{\Gamma\left(\frac{1}{2}(c+f-a+n)-1+z_1\right)(b-z_2)!}{\left[\frac{1}{2}(a-c+f)-z_1\right]!\left[\frac{1}{2}(b+d-f)-z_2\right]!(a+b+n-3-z_1-z_2)!} \\
&\quad\times \frac{\left[\frac{1}{2}(a+b-e)-z_3\right]!\,\Gamma\left(a+b+\frac{1}{2}(d-c-e+n)-1-z_1-z_2-z_3\right)}{\left[\frac{1}{2}(a+b-e)-z_1-z_3\right]!\left[\frac{1}{2}(a+b-e)-z_2-z_3\right]!\,\Gamma\left(e+\frac{1}{2}n+z_3\right)} \\
&\quad\times \frac{\left[\frac{1}{2}(a+b+c+d)+n-3-z_2\right]!\,\Gamma\left(\frac{1}{2}(c-d+e+n)-1+z_3\right)}{\Gamma\left(\frac{1}{2}(b+d+f+n)-z_2\right)\Gamma\left(\frac{1}{2}(a+b-e+n)-1-z_3\right)}
\end{aligned}
\tag{4.4}
$$

where the separate sums are the balanced ${}_4F_3(1)$ series. The total number of terms in (4.4) does not exceed $\frac{1}{6}(r_{11}+1)(r_{11}+2)(2r_{11}+3)$ (but may be surpassed by $(r_{11}+1)(r_{13}+1)$ of (3.6a) or by $(r_{11}+1)(r_{31}+1)$ of (3.6b) or (3.6c)). This expression does not exhibit the usual or Regge-type symmetry of $6j$-symbols of $SO(n)$.

It should be noted that all three summation intervals for the triple sum expressions, derived directly from (2.1) after using diverse expressions for the Racah coefficients (together with Dougall's summation formula [35, 36] of ${}_5F_4(1)$ series) are never restricted by a single parameter.

5. Expansions in terms of Pochhammer symbols and Kampé de Fériet series

Using the parameters $r_{ik} = \beta_i - \alpha_k$ of modified Shelepin's [41] R-array (3.10b) (together with invariant parameters (3.11)) and Pochhammer symbols, we may rewrite the expressions for Regge symmetrical quantities in the following form:

$$
\begin{aligned}
\frac{c^{(\alpha,n)}_{a,b,e;d,c,f}}{(a,b,c,d,e,f)_{[n]}} &= \frac{(-1)^{\alpha_1-\alpha_3}(\alpha_3+n-3)!}{\Gamma^3(n/2)(n-3)!r_{11}!r_{12}!r_{13}!r_{14}!r_{21}!r_{33}!} \\
&\quad\times \Gamma\left[\begin{matrix} r_{22}+\tau, r_{23}+\tau, r_{24}+\tau, r_{32}+\tau, r_{33}+\tau, r_{34}+\tau \\ \alpha_2+\tau+1, \alpha_3+\tau+1, \alpha_4+\tau+1 \end{matrix}\right] \\
&\quad\times \sum_{x_1,x_2} \binom{r_{11}}{x_1}\binom{r_{13}}{x_2}(-1)^{x_1+x_2}(-r_{14}, r_{22}+1, r_{23}+\tau)_{x_1} \\
&\quad\times (-r_{21}, -\alpha_4-\tau, r_{34}+\tau)_{r_{11}-x_1} \\
&\quad\times (r_{24}+\tau, -r_{12}-\tau+1)_{x_2}(-\alpha_2-\tau, r_{32}+\tau)_{r_{13}-x_2} \\
&\quad\times (\beta_2-\beta_1+x_2+1)_{x_1}(-r_{21}-\tau-x_2+1)_{r_{11}-x_1}
\end{aligned}
\tag{5.1a}
$$

$$= \frac{(-1)^{\beta_1-\beta_3}(\alpha_1+n-3)!}{\Gamma^3(n/2)(n-3)!r_{11}!r_{12}!r_{14}!r_{21}!r_{31}!r_{33}!}$$
$$\times \Gamma\left[\begin{matrix} r_{12}+\tau, r_{22}+\tau, r_{23}+\tau, r_{24}+\tau, r_{33}+\tau, r_{34}+\tau \\ \alpha_2+\tau+1, \alpha_3+\tau+1, \alpha_4+\tau+1 \end{matrix}\right]$$
$$\times \sum_{x_1,x_2} \binom{r_{11}}{x_1}\binom{r_{31}}{x_2}(-1)^{x_1+x_2}(-r_{14}, r_{22}+1, r_{23}+\tau)_{x_1}$$
$$\times (-r_{21}, r_{34}+\tau, -\alpha_4-\tau)_{r_{11}-x_1}(-\alpha_2-\tau, -\alpha_3-n+3)_{x_2}$$
$$\times (r_{24}+\tau, \alpha_1+n-2)_{r_{31}-x_2}(r_{34}-x_2+1)_{r_{11}-x_1}$$
$$\times (-r_{34}-r_{11}-\tau+x_2+1)_{x_1} \qquad (5.1b)$$
$$= \frac{(-1)^{\beta_1-\beta_3}(\alpha_1+n-3)!}{\Gamma^3(n/2)(n-3)!r_{11}!r_{12}!r_{21}!r_{31}!r_{33}!r_{34}!}$$
$$\times \Gamma\left[\begin{matrix} r_{22}+\tau, r_{23}+\tau, r_{24}+\tau, r_{32}+\tau, r_{33}+\tau, r_{34}+\tau \\ \alpha_2+\tau+1, \alpha_3+\tau+1, \alpha_4+\tau+1 \end{matrix}\right]$$
$$\times \sum_{x_1,x_2} \binom{r_{11}}{x_1}\binom{r_{31}}{x_2}(-1)^{x_1+x_2}(-r_{12}, -\alpha_3-\tau, -\alpha_4-\tau)_{x_1}$$
$$\times (r_{32}+\tau, r_{22}+1, \alpha_1+n-2)_{r_{11}-x_1}$$
$$\times (r_{23}+\tau, r_{24}+\tau)_{x_2}(-\alpha_2-\tau, -r_{21}-\tau+1)_{r_{31}-x_2}$$
$$\times (-r_{32}-r_{11}-\tau+x_2+1)_{x_1}(r_{32}-x_2+1)_{r_{11}-x_1} \qquad (5.1c)$$

where $\tau = n/2-1$. For products of several gamma functions in the numerator and denominator and Pochhammer symbols (appearing here only in the numerator) we use the notation

$$\Gamma\left[\begin{matrix} a_1, a_2, \ldots, a_A \\ b_1, b_2, \ldots, b_B \end{matrix}\right] = \frac{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_A)}{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_B)} \qquad (5.2)$$

$$(a_1, a_2, \ldots, a_A)_k = (a_1)_k(a_2)_k\ldots(a_A)_k = \frac{\Gamma(a_1+k)\Gamma(a_2+k)\ldots\Gamma(a_A+k)}{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_A)}. \qquad (5.3)$$

Now we may express the terminating double hypergeometric series in (5.1*a*)–(5.1*c*) in terms of special Kampé de Fériet [37] function $F^{1:4;4}_{1:3;3}[{}^{\cdots}_{\cdots}; 1, 1]$, which is defined as follows:

$$F^{1:4;4}_{1:3;3}\left[\begin{matrix} a_1 \\ c_1 \end{matrix} : \begin{matrix} (b) \\ (d) \end{matrix}; \begin{matrix} (b') \\ (d') \end{matrix}; x, y\right] = \sum_{s,t}^{\infty} \frac{(a_1)_{s+t}}{s!t!(c_1)_{s+t}} \frac{\prod_{i=1}^4 (b_i)_s(b'_i)_t}{\prod_{j=1}^3 (d_j)_s(d'_j)_t} x^s y^t \qquad (5.4)$$

and is terminating, because some separate numerator parameters are equal to negative integers: e.g. $b_i = -m$ and $b'_{i'} = -n$ in (5.4), with m and n positive integers. In both cases some of the denominator parameters may be negative integers, but they should be smaller than the parameters responsible for the termination of series. Both separate series are balanced ${}_5F_4(1)$ series with parameters satisfying conditions

$$c_1 - a_1 = 1 + \sum_{i=1}^4 b_i - \sum_{j=1}^3 d_j = 1 + \sum_{i=1}^4 b'_i - \sum_{j=1}^3 d'_j = \tau - 1 = n/2 - 2. \qquad (5.5)$$

Further we denote $r_{jk}+\tau$ by $\hat{r}_{jk}$, $\alpha_k+\tau$ by $\hat{\alpha}_k$ and $\beta_j+\tau$ by $\hat{\beta}_j$ and write the expression for (5.1a) in terms of special Kampé de Fériet series as follows:

$$\begin{aligned}\frac{c^{(\alpha,n)}_{a,b,e;d,c,f}}{(a,b,c,d,e,f)_{[n]}} &= \frac{(\alpha_3+n-3)!}{\Gamma^3(n/2)(n-3)!r_{11}!r_{12}!r_{13}!r_{14}!r_{33}!(\beta_2-\beta_1)!}\\ &\quad\times\Gamma\begin{bmatrix}\hat{r}_{21},\hat{r}_{22},\hat{r}_{23},\hat{r}_{24},\hat{r}_{33},\hat{r}_{34}+r_{11},\hat{r}_{32}+r_{13}\\ \hat{\alpha}_3+1,\hat{\beta}_2-\beta_1,\hat{\alpha}_2-r_{13}+1,\hat{\alpha}_4-r_{11}+1\end{bmatrix}\\ &\quad\times F^{1:4;4}_{1:3;3}\left[\begin{matrix}\beta_2-\beta_1+1\\ \hat{\beta}_2-\beta_1\end{matrix}:\begin{matrix}-r_{11},-r_{14},\hat{r}_{23},r_{22}+1\\ \beta_2-\beta_1+1,\hat{\alpha}_4-r_{11}+1,-\hat{r}_{34}-r_{11}+1\end{matrix};\right.\\ &\qquad\left.\begin{matrix}\hat{r}_{21},\hat{r}_{24},-r_{13},-\hat{r}_{12}+1\\ \beta_2-\beta_1+1,-\hat{r}_{32}-r_{13}+1,\hat{\alpha}_2-r_{13}+1\end{matrix};1,1\right] \qquad (5.6a)\\ &= \frac{(-1)^{\alpha_1-\alpha_3}(\alpha_3+n-3)!(r_{11}+r_{22})!(r_{11}+r_{23})!}{\Gamma^3(n/2)(n-3)!r_{11}!r_{12}!r_{13}!r_{21}!r_{22}!r_{23}!r_{33}!(\alpha_1-\alpha_4)!}\\ &\quad\times\Gamma\begin{bmatrix}\hat{r}_{12},\hat{r}_{22},\hat{r}_{32},\hat{r}_{33},\hat{r}_{34},r_{13}+\hat{r}_{24},r_{11}+\hat{r}_{23}\\ \hat{\alpha}_2+1,\hat{\alpha}_3+1,\hat{\alpha}_4+1,\hat{\alpha}_3-\alpha_2\end{bmatrix}\\ &\quad\times F^{1:4;4}_{1:3;3}\left[\begin{matrix}-r_{11}-\hat{r}_{23}+1\\ -r_{11}-r_{23}\end{matrix}:\begin{matrix}-r_{11},-r_{21},-\hat{\alpha}_4,\hat{r}_{34}\\ -r_{11}-\hat{r}_{23}+1,\alpha_1-\alpha_4+1,-r_{11}-r_{22}\end{matrix};\right.\\ &\qquad\left.\begin{matrix}-r_{23},-r_{13},\hat{r}_{32},-\hat{\alpha}_2\\ -r_{11}-\hat{r}_{23}+1,-r_{13}-\hat{r}_{24}+1,\hat{\alpha}_3-\alpha_2\end{matrix};1,1\right] \qquad (5.6b)\end{aligned}$$

where parameters r_{11} and r_{13} of array (3.10a) are responsible for the termination of series in both cases (and may be replaced by r_{21} and r_{23} in the second (5.6b) case).

Similarly, we write expression for (5.1b) in terms of special Kampé de Fériet series as follows:

$$\begin{aligned}\frac{c^{(\alpha,n)}_{a,b,e;d,c,f}}{(a,b,c,d,e,f)_{[n]}} &= \frac{(-1)^{\beta_1-\beta_3}(r_{34}+r_{11})!(\beta_3+n-3)!}{\Gamma^3(n/2)(n-3)!r_{11}!r_{12}!r_{14}!r_{31}!r_{33}!r_{34}!(\beta_2-\beta_1)!}\\ &\quad\times\Gamma\begin{bmatrix}\hat{r}_{12},\hat{r}_{22},\hat{r}_{23},\hat{r}_{33},\hat{r}_{24}+r_{31},\hat{r}_{34}+r_{11}\\ \hat{\alpha}_2+1,\hat{\alpha}_3+1,\hat{\alpha}_4-r_{11}+1\end{bmatrix}\\ &\quad\times F^{1:4;4}_{1:3;3}\left[\begin{matrix}-\hat{r}_{34}-r_{11}+1\\ -r_{34}-r_{11}\end{matrix}:\begin{matrix}-r_{11},-r_{14},\hat{r}_{23},r_{22}+1\\ -\hat{r}_{34}-r_{11}+1,\beta_2-\beta_1+1,\hat{\alpha}_4-r_{11}+1\end{matrix};\right.\\ &\qquad\left.\begin{matrix}-r_{34},-r_{31},-\hat{\alpha}_2,-\alpha_3-n+3\\ -\hat{r}_{14}-r_{31}+1,-\hat{r}_{24}-r_{31}+1,-\beta_3-n+3\end{matrix};1,1\right] \qquad (5.7a)\\ &= \frac{(\alpha_1+n-3)!(\alpha_3+n-3)!(r_{11}+r_{22})!}{\Gamma^3(n/2)(n-3)!r_{11}!r_{12}!r_{21}!r_{22}!r_{31}!r_{33}!(\alpha_3-r_{31}+n-3)!}\\ &\quad\times\frac{1}{(\alpha_1-\alpha_4)!}\Gamma\begin{bmatrix}\hat{r}_{12},\hat{r}_{14},\hat{r}_{22},\hat{r}_{24},\hat{r}_{33},\hat{r}_{34},r_{11}+\hat{r}_{23}\\ \hat{\alpha}_3+1,\hat{\alpha}_4+1,\hat{\alpha}_1-\alpha_4,\hat{\alpha}_2-r_{31}+1\end{bmatrix}\\ &\quad\times F^{1:4;4}_{1:3;3}\left[\begin{matrix}\alpha_1-\alpha_4+1\\ \hat{\alpha}_1-\alpha_4\end{matrix}:\begin{matrix}-r_{11},-r_{21},\hat{r}_{34},-\hat{\alpha}_4\\ \alpha_1-\alpha_4+1,-r_{11}-\hat{r}_{23}+1,-r_{11}-r_{22}\end{matrix};\right.\\ &\qquad\left.\begin{matrix}\hat{r}_{14},\hat{r}_{24},-r_{31},\alpha_1+n-2\\ \alpha_1-\alpha_4+1,\hat{\alpha}_2-r_{31}+1,\alpha_3-r_{31}+n-2\end{matrix};1,1\right] \qquad (5.7b)\end{aligned}$$

where parameters r_{11} and r_{31} are responsible for the termination of series in both cases (and may be replaced by r_{14} and r_{34} in the first (5.7a) case).

Finally, we write expression for (5.1c) in terms of special Kampé de Fériet series as follows:

$$\frac{c^{(\alpha,n)}_{a,b,e;d,c,f}}{(a,b,c,d,e,f)_{[n]}} = \frac{(-1)^{\beta_1-\beta_3}(\beta_1+n-3)!(r_{11}+r_{22})!(r_{11}+r_{32})!}{\Gamma^3(n/2)(n-3)!r_{11}!r_{12}!r_{21}!r_{22}!r_{31}!r_{32}!r_{33}!r_{34}!}$$
$$\times\Gamma\left[\begin{matrix}\hat{r}_{21},\hat{r}_{22},\hat{r}_{23},\hat{r}_{24},\hat{r}_{33},\hat{r}_{34},\hat{r}_{32}+r_{11}\\ \hat{\alpha}_3+1,\hat{\alpha}_4+1,\hat{\alpha}_2-r_{31}+1,\hat{\beta}_2-\beta_3\end{matrix}\right]$$
$$\times F^{1:4;4}_{1:3;3}\left[\begin{matrix}-\hat{r}_{32}-r_{11}+1\\ -r_{32}-r_{11}\end{matrix}:\begin{matrix}-r_{11},-r_{12},-\hat{\alpha}_3,-\hat{\alpha}_4\\ -\hat{r}_{32}-r_{11}+1,-\beta_1-n+3,-r_{11}-r_{22}\end{matrix};\right.$$
$$\left.\begin{matrix}-r_{32},-r_{31},\hat{r}_{24},\hat{r}_{23}\\ -\hat{r}_{32}-r_{11}+1,\hat{\alpha}_2-r_{31}+1,\hat{\beta}_2-\beta_3\end{matrix};1,1\right] \tag{5.8a}$$
$$=\frac{(\alpha_1+n-3)!}{\Gamma^3(n/2)(n-3)!r_{11}!r_{21}!r_{31}!r_{33}!r_{34}!(\alpha_1-\alpha_2)!}$$
$$\times\Gamma\left[\begin{matrix}\hat{r}_{12},\hat{r}_{22},\hat{r}_{32},\hat{r}_{33},\hat{r}_{34},\hat{r}_{23}+r_{31},\hat{r}_{24}+r_{31}\\ \hat{\alpha}_2+1,\hat{\alpha}_3-r_{11}+1,\hat{\alpha}_4-r_{11}+1,\hat{\alpha}_1-\alpha_2\end{matrix}\right]$$
$$\times F^{1:4;4}_{1:3;3}\left[\begin{matrix}\alpha_1-\alpha_2+1\\ \hat{\alpha}_1-\alpha_2\end{matrix}:\begin{matrix}-r_{11},\hat{r}_{32},\alpha_1+n-2,r_{22}+1\\ \alpha_1-\alpha_2+1,\hat{\alpha}_3-r_{11}+1,\hat{\alpha}_4-r_{11}+1\end{matrix};\right.$$
$$\left.\begin{matrix}\hat{r}_{12},-r_{31},-\hat{\alpha}_2,-\hat{r}_{21}+1\\ \alpha_1-\alpha_2+1,-\hat{r}_{24}-r_{31}+1,-\hat{r}_{23}-r_{31}+1\end{matrix};1,1\right] \tag{5.8b}$$

where parameters r_{11} and r_{31} are responsible for the termination of series in both cases (and may be replaced by r_{12} and r_{32} in the first (5.8a) case). However, the possible indefiniteness (appearing, e.g., with negative integer arguments of $(\beta_2-\beta_1)!$ or $\Gamma(\hat{\alpha}_1-\alpha_2)$ and never troubling in expressions (5.1a)–(5.1c)) should be kept in mind in expressions (5.6a)–(5.8b) in terms of special Kampé de Fériet series.

There are the definite linear dependences

$$\begin{aligned} &a_1=d_1=d'_1=d_j+d'_j-1 && (j=2,3)\\ &c_1=b_4+b'_4-n+4=b_i+b'_i && (i=1,2,3)\end{aligned} \tag{5.9a}$$

between parameters of each special Kampé de Fériet series in (5.6a), (5.7a) and (5.8b), as well as the relations

$$\begin{aligned} &a_1=d_1=d'_1=d_2+d'_2-1=d_3+d'_3-n+3\\ &c_1=b_i+b'_i \qquad (i=1,\ldots,4)\end{aligned} \tag{5.9b}$$

between parameters of each special Kampé de Fériet series in (5.6b), (5.7b) and (5.8a). Of course, parameters $a_1=d_1=d'_1$ (which are integers depending on the distance between the rows or columns of Shelepin's array in (5.6a) or in (5.7b) and (5.8b), respectively) are never responsible for the termination of series. There is absent any correlation between the types of dependences (5.9a)–(5.9b) and the types of parameters a_1 and c_1, the last being non-positive integers in (5.6b), (5.7a) and (5.8a). Nevertheless, expressions (5.6a) and (5.7a) (as well as (5.6b) and (5.7b)) are mutually related with respect to the substitution (hook reflection)

$$d\to -d-n+2 \tag{5.10a}$$

leaving invariant the dimension $d_d^{(n)}$ and character of irrep d of $SO(n)$, when (5.8b) is invariant under this substitution, which corresponds to the transposition $\hat{r}_{32} \leftrightarrow \alpha_1 + n - 2$ (together with $-r_{31} \leftrightarrow -\hat{\alpha}_2$) of parameters. Otherwise, expressions (5.6a) and (5.8b) (as well as (5.6b) and (5.8a)) are mutually related with respect to the hook reflections

$$c \to -c - n + 2 \qquad d \to -d - n + 2 \qquad f \to -f - n + 2, \tag{5.10b}$$

leaving invariant the dimensions and characters of irreps c, d and f.

The transposition $r_{11} \leftrightarrow r_{14}$ (together with $\hat{r}_{24} \leftrightarrow \hat{r}_{21}$) in (5.6$a$) gives Regge symmetry (3.8), as well as $r_{11} \leftrightarrow r_{14}$ (together with $r_{31} \leftrightarrow r_{34}$) in (5.7$a$). The transposition $r_{11} \leftrightarrow r_{21}$ (together with $r_{13} \leftrightarrow r_{23}$) in (5.6$b$) gives Regge symmetry

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{SO(n)} = \begin{Bmatrix} a & s_1 - c & s_1 - f \\ d & s_1 - b & s_1 - e \end{Bmatrix}_{SO(n)} \tag{5.11}$$

which corresponds to the transposition of rows in array (3.10b), as well as the transposition $r_{11} \leftrightarrow r_{21}$ (together with $\hat{r}_{14} \leftrightarrow \hat{r}_{24}$) in (5.7$b$). Otherwise, transpositions $r_{11} \leftrightarrow r_{12}$ (together with $r_{31} \leftrightarrow r_{32}$) and $\hat{\alpha}_3 \leftrightarrow \hat{\alpha}_4$ (together with $\hat{r}_{23} \leftrightarrow \hat{r}_{24}$) are the generators of symmetries (3.9a) and (3.9b) for (5.8a).

Further, the transposition $-r_{14} \leftrightarrow \hat{r}_{23}$ (together with $-r_{13} \leftrightarrow \hat{r}_{24}$ in (5.6a) or with $r_{31} \leftrightarrow \hat{\alpha}_2$ in (5.7a), respectively), the transpositions $r_{11} \leftrightarrow \hat{\alpha}_4$ (together with $-r_{23} \leftrightarrow \hat{r}_{32}$ or with $\hat{r}_{14} \leftrightarrow \alpha_1 + n - 2$) and $-r_{21} \leftrightarrow \hat{r}_{34}$ (together with $r_{13} \leftrightarrow \hat{\alpha}_2$ or with $\hat{r}_{24} \leftrightarrow -r_{31}$) in (5.6$b$) or (5.7$b$), respectively, and the interchange of r_{11} and $\hat{\alpha}_4$ (together with $-r_{32} \leftrightarrow \hat{r}_{23}$) or the interchange of r_{12} and $\hat{\alpha}_3$ (together with $-r_{31} \leftrightarrow \hat{r}_{24}$) in (5.8$a$) correspond to some hook reflections. The remaining transpositions from 3! or 4!, leaving invariant dependences (5.9a) or (5.9b), correspond to compositions of some Regge symmetries and hook reflections, as well as the interchanges of the sets b_1, b_2, b_3, b_4; d_2, d_3 and b_1', b_2', b_3', b_4'; d_2', d_3' in (5.4).

6. Concluding remarks

In this paper, we considered the recoupling coefficients and $6j$-symbols of the orthogonal $SO(n)$ groups for all six representations corresponding to the spherical or hyperspherical harmonics of these groups. The corrected triple sum expression of Ališauskas [16] (which was previously derived from the fourfold sum expression of [16], related to a later expression of [19]) as an expansion in terms of the $6j$ coefficients of $SU(2)$, with possible multiples of $1/4$ parameters for odd n, here has been rearranged into three different double hypergeometric series of Kampé de Fériet $F^{1:4;4}_{1:3;3}$ type with the moderate (2×2) symmetry which (together with the hidden usual symmetry) nevertheless is resolving for all 144 Regge-type symmetries. The different double sum expressions are mutually related with respect to the substitutions, generalizing the 'mirror reflection' symmetry $j \to -j - 1$ of the angular momentum theory [29]. Note that more general double $F^{1:4;4}_{1:3;3}$ series (balanced by condition (5.5) parameters and four (from 8) separate relations of the type (5.9a) or (5.9b)) appeared as the doubly stretched $12j$ coefficients of the second kind [34] for $SU(2)$ as presented by expressions (2.5a)–(2.5c) (with $z_1 = z_2 = 0$) for $j_1 = k_1 - l_1 = l_2 - k_3$, or by expression (2.9) for $k_1 = j_1 + l_1 = j_2 + l_3$, although our $6j$-symbols of $SO(n)$ cannot be reduced to special $12j$ coefficients of $SU(2)$.

However, two non-positive integer parameters are necessary for termination of these double series, in contrast with special double hypergeometric series $F^{1:2,2}_{1:1,1}$ of Kampé de Fériet type [33, 49, 50] which correspond to the stretched $9j$ coefficients of $SU(2)$, and may terminate for fixed single integer non-positive parameter, restricting all summation parameters. A single integer non-positive parameter is sufficient to restrict all summation parameters only in the triple sum expression, derived in section 4. (These parameters appear in the double sums over

z_1, z_3 and over z_2, z_3, which again may be treated as the double hypergeometric series $F^{1:3,3}_{1:2,2}$ of Kampé de Fériet type.)

Related recoupling coefficients of the symplectic groups $Sp(2n)$ with all six irreps antisymmetric are also given in the appendix as the double series which correspond to the orthogonal groups $SO(-2n)$ of negative dimension, with richer restriction structure. These $Sp(2n)$ invariants never form the complete recoupling matrices, as well as the $SO(n)$ invariants considered in this paper as the main objects. Note the essential difference between our $6j$-symbols and the $6j$-symbols of the orthogonal $SO(n)$ groups for all six irreps antisymmetric and $6j$-symbols of the symplectic groups $Sp(2n)$ with all six irreps symmetric as expressed [3, 5, 22] in terms of the hypergeometric (single) ${}_4F_3(1)$ series with all 144 Regge-type symmetries visible immediately. Alternating single sums appear in the $Sp(2n)$ case, but these ${}_4F_3(1)$ series are balanced (Saalschützian) only in the $Sp(2)$ or $SU(2)$ case, when these $6j$-symbols also form the complete recoupling matrices.

Appendix. Recoupling coefficients for antisymmetric representations of *Sp*(2*n*)

As demonstrated by Judd *et al* [7] the recoupling function[1]

$$U\begin{pmatrix} A & B & E \\ D & C & F \end{pmatrix}_{Sp(2n)} (AM_A, FM_F|CM_C) = \sum_{M_B,M_D,M_E} (DM_D, BM_B|FM_F) \times (AM_A, BM_B|EM_E)(EM_E, DM_D|CM_C) \tag{A.1}$$

of the symplectic group $Sp(2n)$ for all six antisymmetric representations $\langle 1^\nu\rangle$ ($\nu \leqslant n$), with dimension (cf [51])

$$d^{\langle 2n\rangle}_{\langle 1^\nu\rangle} = \frac{2(2n+1)!(n-\nu+1)}{\nu!(2n-\nu+2)!} \tag{A.2}$$

may be expressed as an analytical continuation of the recoupling coefficients for symmetric representations of the orthogonal group of the negative rank $SO(-2n)$, in accordance with [3, 7, 22] (cf also [42, 43]) as follows:

$$U\begin{pmatrix} \langle 1^a\rangle & \langle 1^b\rangle & \langle 1^e\rangle \\ \langle 1^d\rangle & \langle 1^c\rangle & \langle 1^f\rangle \end{pmatrix}_{Sp(2n)} = (-1)^{\chi_{12}} \left(d_e^{(-2n)} d_f^{(-2n)}\right)^{1/2} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_{SO(-2n)} \tag{A.3}$$

(ignoring possible phase factor $(-1)^\chi$).

It is most convenient to perform the analytical continuation of expressions (5.1*a*)–(5.1*c*), together with (3.4*b*) and (2.4), using the same integer parameters $r_{ik} = \beta_i - \alpha_k$ ($i = 1, 2, 3$; $j = 1, \ldots, 4$) of array (3.10*b*), or the most symmetric parametrization (3.11). Hence replacing n by $-2n$ and τ by $-n-1$ we write the following three expressions for the recoupling coefficients of the symplectic group:

$$U\begin{pmatrix} \langle 1^a\rangle & \langle 1^b\rangle & \langle 1^e\rangle \\ \langle 1^d\rangle & \langle 1^c\rangle & \langle 1^f\rangle \end{pmatrix}_{Sp(2n)} = \frac{(-1)^{\chi_{12}}\left(d^{\langle 2n\rangle}_{\langle 1^e\rangle} d^{\langle 2n\rangle}_{\langle 1^f\rangle}\right)^{1/2} Nn!^3(2n+2)!}{(2n+2-\alpha_3)!r_{11}!r_{12}!r_{13}!r_{14}!r_{21}!r_{33}!(n-r_{22}+1)!}$$
$$\times \frac{(n-\alpha_2)!(n-\alpha_3)!(n-\alpha_4)!}{(n-r_{23}+1)!(n-r_{24}+1)!(n-r_{32}+1)!(n-r_{33}+1)!(n-r_{34}+1)!}$$
$$\times \sum_{x_1,x_2} \binom{r_{11}}{x_1}\binom{r_{13}}{x_2}(-1)^{x_1+x_2}(-r_{14}, r_{22}+1, r_{23}-n-1)_{x_1}$$

[1] Which here is defined in a formal way with arbitrary labels of the basis states, without the need for internal and external multiplicity labels.

$$\times(-r_{21},-\alpha_4+n+1,r_{34}-n-1)_{r_{11}-x_1}$$
$$\times(r_{24}-n-1,-r_{12}+n+2)_{x_2}(-\alpha_2+n+1,r_{32}-n-1)_{r_{13}-x_2}$$
$$\times(\beta_2-\beta_1+x_2+1)_{x_1}(-r_{21}+n+2-x_2)_{r_{11}-x_1} \qquad (A.4a)$$

$$=\frac{(-1)^{\chi_{12}}\left(d^{\langle 2n\rangle}_{\langle 1^e\rangle}d^{\langle 2n\rangle}_{\langle 1^f\rangle}\right)^{1/2}Nn!^3(2n+2)!}{(2n-\alpha_1+2)!r_{11}!r_{12}!r_{14}!r_{21}!r_{31}!r_{33}!(n-r_{12}+1)!(n-r_{22}+1)!}$$
$$\times\frac{(n-\alpha_2)!(n-\alpha_3)!(n-\alpha_4)!}{(n-r_{23}+1)!(n-r_{24}+1)!(n-r_{33}+1)!(n-r_{34}+1)!}$$
$$\times\sum_{x_1,x_2}\binom{r_{11}}{x_1}\binom{r_{31}}{x_2}(-1)^{x_1+x_2}(-r_{14},r_{22}+1,r_{23}-n-1)_{x_1}$$
$$\times(-r_{21},r_{34}-n-1,-\alpha_4+n+1)_{r_{11}-x_1}(-\alpha_2+n+1,-\alpha_3+2n+3)_{x_2}$$
$$\times(r_{24}-n-1,\alpha_1-2n-2)_{r_{31}-x_2}(r_{34}-x_2+1)_{r_{11}-x_1}$$
$$\times(-r_{34}-r_{11}+n+x_2+2)_{x_1} \qquad (A.4b)$$

$$=\frac{(-1)^{\chi_{12}+\beta_1-\beta_3}\left(d^{\langle 2n\rangle}_{\langle 1^e\rangle}d^{\langle 2n\rangle}_{\langle 1^f\rangle}\right)^{1/2}Nn!^3(2n+2)!}{(2n-\alpha_1+2)!r_{11}!r_{12}!r_{21}!r_{31}!r_{33}!r_{34}!(n-r_{22}+1)!}$$
$$\times\frac{(n-\alpha_2)!(n-\alpha_3)!(n-\alpha_4)!}{(n-r_{23}+1)!(n-r_{24}+1)!(n-r_{33}+1)!(n-r_{34}+1)!}$$
$$\times\sum_{x_1,x_2}\binom{r_{11}}{x_1}\binom{r_{31}}{x_2}(-1)^{x_1+x_2}(-r_{12},-\alpha_3+n+1,-\alpha_4+n+1)_{x_1}$$
$$\times(r_{32}-n-1,r_{22}+1,\alpha_1-2n-2)_{r_{11}-x_1}$$
$$\times(r_{23}-n-1,r_{24}-n-1)_{x_2}(-\alpha_2+n+1,-r_{21}+n+2)_{r_{31}-x_2}$$
$$\times(-r_{32}-r_{11}+n+x_2+2)_{x_1}(r_{32}-x_2+1)_{r_{11}-x_1} \qquad (A.4c)$$

where actually some ratios of gamma functions $\Gamma(-y)/\Gamma(-x)$ turned into the ratios of factorials $(-1)^{x-y}x!/y!$ only in the factor under the square root of (3.4*b*) (together with (2.4)) which absolute value appeared as

$$N=\frac{\left[\prod_{i=1}^{3}\prod_{k=1}^{4}r_{ik}!(n+1-r_{ik})!\right]^{1/2}}{(2n+2)!^2n!^4}\left[\prod_{k=1}^{4}\frac{(2n+2-\alpha_k)!}{(n-\alpha_k)!}\right]^{1/2}$$

in the dimension factors and in the factors before the summation sign of (5.1*a*)–(5.1*c*). Intervals of summation in (A.4*a*)–(A.4*c*) are restricted not only by conditions (3.7*a*)–(3.7*d*) (e.g., $r_{11}\geqslant 0$ and $r_{13}\geqslant 0$, or $r_{31}\geqslant 0$) but also by conditions of the type $n+1-r_{ik}\geqslant 0$. Therefore it may be convenient to write (A.4*a*)–(A.4*c*) as factorial series, with summation intervals determined by non-negative arguments of the denominator factorials. Some factorials before the sum signs cancel. All the separate sums in such a new version of (A.4*a*)–(A.4*c*) are alternating, with the exception of the sum over x_1 in (A.4*c*). Of course, the recoupling coefficients of the symplectic group $Sp(2n)$ with all six antisymmetric irreps vanish unless $n-\alpha_k\geqslant 0$. The number of different recoupling coefficients of $Sp(2n)$ for fixed n is finite, in contrast with the number of $6j$-symbols of $SO(n)$, considered in this paper.

The symmetry properties of the recoupling functions of the symplectic group $Sp(2n)$ may be more complicated, e.g.

$$\left(d^{\langle 2n\rangle}_{\langle 1^b\rangle}d^{\langle 2n\rangle}_{\langle 1^c\rangle}\right)^{-1/2} U\begin{pmatrix}\langle 1^a\rangle & \langle 1^e\rangle & \langle 1^b\rangle\\ \langle 1^d\rangle & \langle 1^f\rangle & \langle 1^c\rangle\end{pmatrix}_{Sp(2n)} = (-1)^{\chi_{13}-\chi_{12}+(b+c-e-f)/2} \times \left(d^{\langle 2n\rangle}_{\langle 1^e\rangle}d^{\langle 2n\rangle}_{\langle 1^f\rangle}\right)^{-1/2} U\begin{pmatrix}\langle 1^a\rangle & \langle 1^b\rangle & \langle 1^e\rangle\\ \langle 1^d\rangle & \langle 1^c\rangle & \langle 1^f\rangle\end{pmatrix}_{Sp(2n)}. \qquad (A.5)$$

The $6j$-symbols of $Sp(2n)$ in the case of all antisymmetric irreps are real and invariant under all permutations of the type (2.5), if we choose $\chi_{13} = \beta_2$, $\chi_{12} = \beta_1$ as defined by (3.11).

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